Lecture 17. Principles of multiple scattering in the atmosphere. Radiative-transfer equation for solar radiation in a plane-parallel atmosphere.

Objectives:

1. Concepts of the direct and diffuse (scattered) solar radiation.
2. Source function and a radiative transfer equation for the diffuse solar radiation.
3. Single scattering approximation.
4. Legendre polynomial expansion of the scattering phase function.

Required reading:

L02: 3.4, 6.1, Appendix E

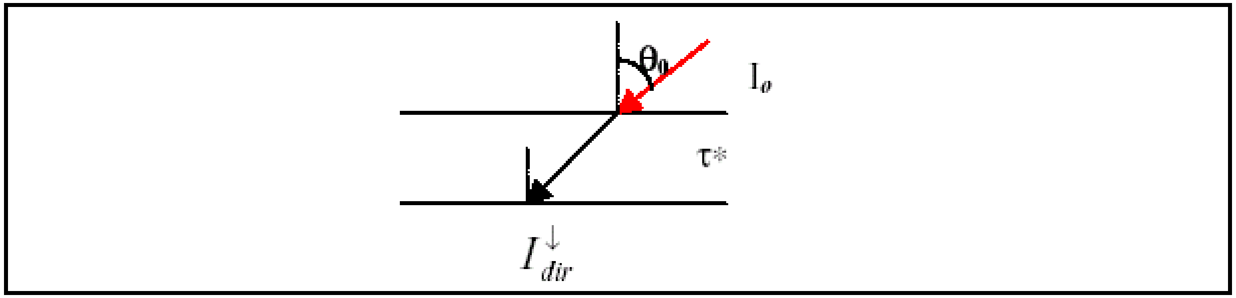
1. Concepts of the direct and diffuse solar radiation.

- The solar radiation field is traditionally considered as a sum of two distinctly different components: **direct** and **diffuse:** $I = I_{dir} + I_{dif}$

Direct solar radiation is a part of solar radiation filed that has survived the extinction passing a layer with optical depth τ^* and it obeys the Beer-Bouguer-Lambert (extinction) law:

$$\boxed{I_{dir}^{\downarrow} = I_0 \exp(-\tau^* / \mu_0)} \qquad [17.1]$$

where I_0 is the solar intensity at a given wavelengths at the top of the atmosphere and μ_0 is a cosine of the solar zenith angle θ_0 ($\mu_0 = cos(\theta_0)$).

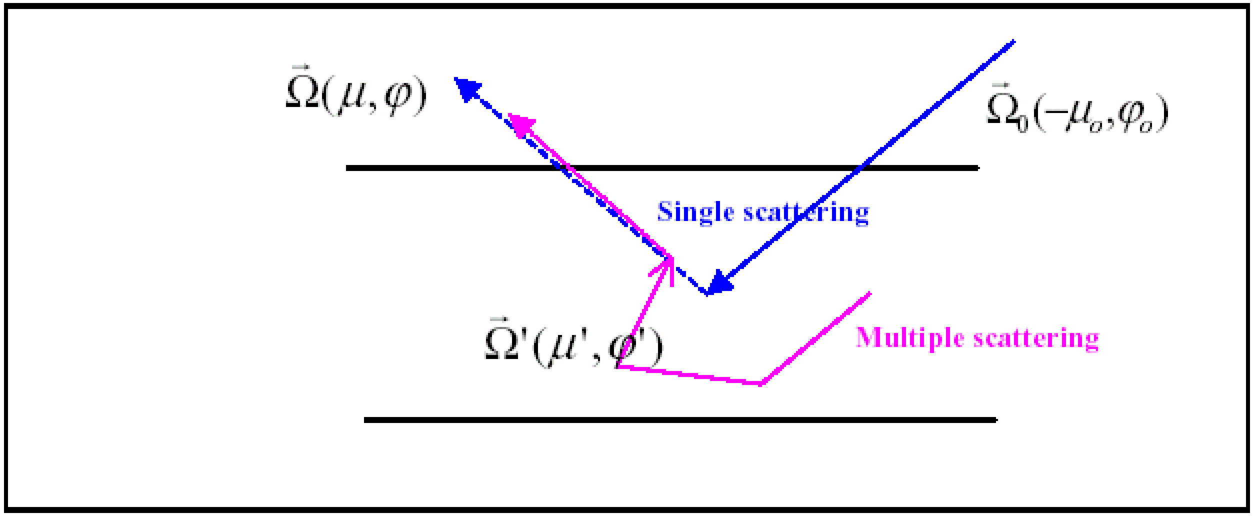


The **direct solar flux** is

$$\boxed{F_{dir}^{\downarrow} = \mu_0 F_0 \exp(-\tau^* / \mu_0)} \qquad [17.2]$$

2.Source function and a radiative transfer equation for the diffuse solar radiation.

Diffuse radiation arises from the light that undergoes one scattering event (**single scattering**) or many (**multiple scattering**).



Recall Lectures 2- 3 where we have defined the source function

$$J_\lambda = (\mathbf{j}_{\lambda,\ \text{thermal}} + \mathbf{j}_{\lambda,\ \text{scattering}}) / \beta_{e,\lambda}$$

where $\mathbf{j}_{\lambda,\ \text{thermal}}$ is the **thermal emission** ($j_{\lambda,thermal} = \beta_{a,\lambda} B_\lambda(T)$)

and $\mathbf{j}_{\lambda,\ \text{scattering}}$ is the re-radiation from multiple scattering.

Using the volume scattering coefficient $\beta_{s,\lambda}$ and the phase function $P(\mu, \varphi, \mu', \varphi')$, we have

$$\boxed{j_{\lambda,scattering}(\vec{\Omega}) = \frac{\beta_{s,\lambda}}{4\pi} \int_{\Omega'} I(\vec{\Omega}') P(\vec{\Omega}, \vec{\Omega}') d\Omega'} \qquad [17.3]$$

NOTE: Recall the **scattering phase function** $P(\mu, \varphi, \mu', \varphi')$ (i.e., the element of the scattering matrix P_{11}) represents the angular distribution of scattered energy as a function of direction. By the definition (see Lecture13), it is normalized as

$$\frac{1}{4\pi} \int_{\Omega} P(\Theta) d\Omega = 1$$

where Θ is the scattering angle

$$\cos(\Theta) = \cos(\theta')\cos(\theta) + \sin(\theta')\sin(\theta)\cos(\varphi'-\varphi) = \mu'\mu + (1-\mu'^2)^{1/2}(1-\mu^2)^{1/2}\cos(\varphi'-\varphi)$$

Scattering of the direct beam is the *source* of diffuse radiation:

$$J_{dif} = \frac{\omega}{4\pi} P(\mu, \phi; -\mu_0, \phi_0 + \pi)\ S_0 e^{-\tau/\mu_0}$$

The boundary condition for diffuse radiation is $I(\infty, \mu, \phi) = 0$ for $\mu < 0$.

Thus the **source function for diffuse solar radiation** may be written as two components

$$\boxed{\begin{aligned} J(\tau,\mu,\varphi) &= \frac{\omega_0}{4\pi}\int_0^{2\pi}\int_{-1}^{1} I(\tau,\mu',\varphi')P(\mu,\varphi,\mu',\varphi')d\mu' d\varphi' + \\ &+ \frac{\omega_0}{4\pi}F_0 P(\mu,\varphi,-\mu_0,\varphi_0)\exp(-\tau/\mu_0) \end{aligned}} \quad [17.4]$$

where the ω_0 is the single scattering albedo and ***P*** is the scattering phase function.

NOTE: In Eq.[17.4], the first term on the right-hand side shows that the phase function redirects the incoming intensity in the direction (μ',φ') to the direction (μ, φ), and the integrals account for all possible scattering events within the 4π solid angle.

- The **source function for scattering** Eq.[17.4] is more complicated than a thermal source function:
  - (i) It involves conditions throughout the atmosphere, while the thermal source function depends on local conditions only;
  - (ii) The phase function ***P***(μ, φ, μ',φ') may be a very complex function of the directions (and, in general, state of polarization).

Plane-Parallel Solar Radiative Transfer Equation

Mostly we will ignore horizontal variability (assume a plane-parallel atmosphere) and omit thermal emission in the shortwave.

The monochromatic solar radiative transfer equation is then

$$\mu \frac{dI(\mu,\phi)}{dz} = -\beta \left[I(\mu,\phi) - \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} P(\Theta)\; I(\mu',\phi')\; d\mu' d\phi' \right]$$

The first term on right is the extinction and the second is the scattering source.

Usually the phase function depends only on the scattering angle Θ:

$$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi' - \phi)$$

Often we use optical depth as the vertical coordinate:

$$\mu \frac{dI(\mu,\phi)}{d\tau} = I(\mu,\phi) - J(\mu,\phi)$$

Recall the radiative transfer equation defined in Lecture 2 for a plane-parallel atmosphere

$$\mu \frac{dI_\lambda(\tau;\mu;\varphi)}{d\tau} = I_\lambda(\tau;\mu;\varphi) - J_\lambda(\tau;\mu;\varphi)$$

Thus, using the source function for scattering, we can write the **radiative transfer equation for the diffuse radiation** as (omitting the subscript *dif* in ***I***)

$$\boxed{\mu \frac{dI(\tau,\vec{\Omega})}{d\tau} = I(\tau,\vec{\Omega}) - \frac{\omega_0}{4\pi}\int_{4\pi} I(\tau,\vec{\Omega}')P(\vec{\Omega},\vec{\Omega}')d\Omega' - \frac{\omega_0}{4\pi} F_0 P(\vec{\Omega},-\vec{\Omega}_0)\exp(-\tau/\mu_0)} \quad [17.5]$$

NOTE: Eq.[17.5] is an integro-differential equation. To solve Eq.[17.5], one needs to know the scattering coefficient $\boldsymbol{\beta_{s,\lambda}}$, absorption coefficient $\boldsymbol{\beta_{a,\lambda}}$ and scattering phase function ***P*(μ, φ, μ',φ')** as a function of wavelength in each atmospheric layer.

Eq.[17.5] can be simplified if there is no dependency on the azimuth angle.

For azimuthally independent case, we may define the phase function as

$$P(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} P(\mu,\varphi,\mu',\varphi')d\varphi' \quad [17.6]$$

Using Eq.[17.6], we may write **the azimuthally independent radiative transfer equation for the diffuse radiation**

$$\boxed{\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\int_{-1}^{1} I(\tau,\mu')P(\mu,\mu')d\mu' - \frac{\omega_0}{4\pi} F_0 P(\mu,-\mu_0)\exp(-\tau/\mu_0)} \quad [17.7]$$

- **To find a solution of the radiative transfer equation for diffuse radiation** (i.e., to solve Eq.[17.5]) , various approximate and “exact” techniques have been developed:

  Approximate methods:

  i) Single scattering approximations (this lecture)

  ii) Two-stream approximations (Lecture 18)

  iii) Eddington and Delta- Eddington approximations (Lecture 18)

  “Exact” methods:

  i) Discrete-ordinate technique (Lecture 20)

  ii) Adding-doubling technique (Lecture 21)

  iii) Monte-Carlo technique (Lecture 22)

3. Single scattering approximation.

If light has been scattered only once, the source function from Eq.[17.3] becomes

$$J(\tau,\mu,\varphi)=\frac{\omega_0}{4\pi}F_0P(\mu,\varphi,-\mu_0,\varphi_0)\exp(-\tau/\mu_0) \qquad [17.8]$$

and using the solution (derived in Lecture 2) of the radiation transfer in a plane-parallel atmosphere bounded by on two sides at τ=0 and τ=τ*:

for upward intensity (reflected)

$$I_\lambda^\uparrow(\tau;\mu;\varphi) = I_\lambda^\uparrow(\tau^*;\mu;\varphi)\exp\left(-\frac{\tau^*-\tau}{\mu}\right)$$

$$+\frac{1}{\mu}\int_\tau^{\tau^*}\exp\left(-\frac{\tau'-\tau}{\mu}\right)J_\lambda^\uparrow(\tau';\mu;\varphi)\,d\tau'$$

and downward intensity (transmitted)

$$I_\lambda^\downarrow(\tau,-\mu,\varphi) = I_\lambda^\downarrow(0,-\mu,\varphi)\exp\left(-\frac{\tau}{\mu}\right)$$

$$+\frac{1}{\mu}\int_0^{\tau}\exp\left(-\frac{\tau-\tau'}{\mu}\right)J_\lambda^\downarrow(\tau',-\mu,\varphi)\,d\tau'$$

we can write **the solution for diffuse radiation in a single scattering approximation** as

$$I_{\lambda}^{\uparrow}(\tau;\mu;\varphi) = I_{\lambda}^{\uparrow}(\tau^*,\mu,\varphi)\exp\left(-\frac{\tau^*-\tau}{\mu}\right)$$
$$+\frac{1}{\mu}\frac{\omega_0}{4\pi}F_0 P(\mu,\varphi,-\mu_0,\varphi_0)\int_{\tau}^{\tau^*}\exp\left(-\left[\frac{\tau'-\tau}{\mu}+\frac{\tau'}{\mu_0}\right]\right)d\tau' \qquad [17.9a]$$

$$I_{\lambda}^{\downarrow}(\tau;-\mu;\varphi) = I_{\lambda}^{\downarrow}(0,-\mu,\varphi)\exp\left(-\frac{\tau}{\mu}\right)$$
$$+\frac{1}{\mu}\frac{\omega_0}{4\pi}F_0 P(-\mu,\varphi,-\mu_0,\varphi_0)\int_{0}^{\tau}\exp\left(-\left[\frac{\tau'-\tau}{\mu}+\frac{\tau'}{\mu_0}\right]\right)d\tau' \qquad [17.9b]$$

Assuming that there is no diffuse downward radiation at the top of the atmosphere

$$I^{\downarrow}(0,-\mu,\varphi)=0$$

and no upward diffuse radiation at the surface (i.e., no reflection from the surface)

$$I^{\uparrow}(\tau^*,\mu,\varphi)=0 \qquad [17.10]$$

Then from Eq.[17.9a,b] for finite atmosphere of the optical depth τ=τ_*, we have the

reflected and transmitted diffuse intensities

$$I_\lambda^\uparrow(0,\mu,\varphi) = \frac{\omega_0\mu_0 F_0}{4\pi(\mu+\mu_0)} P(\mu,\varphi,-\mu_0,\varphi_0)\left[1-\exp\left(-\tau^*(\frac{1}{\mu}+\frac{1}{\mu_0})\right)\right] \qquad [17.11]$$

and for μ is NOT equaled to μ_0

$$I_\lambda^\downarrow(\tau^*,-\mu,\varphi) = \frac{\omega_0\mu_0 F_0}{4\pi(\mu-\mu_0)} P(-\mu,\varphi,-\mu_0,\varphi_0)\left[\exp\left(-\frac{\tau^*}{\mu}\right)-\exp\left(-\frac{\tau^*}{\mu_0}\right)\right] \qquad [17.12]$$

and for μ=μ_0

$$I_\lambda^\downarrow(\tau^*,-\mu,\varphi) = \frac{\omega_0\tau^* F_0}{4\pi\mu_0} P(-\mu_0,\varphi_0,-\mu_0,\varphi_0)\left[\exp\left(-\frac{\tau^*}{\mu_0}\right)\right] \qquad [17.13]$$

- For the single scattering approximation, the diffuse intensities are directly proportional to the phase function.

NOTE: the single scattering approximation is valid for the optically thin atmosphere (i.e., small optical depth).

First Order Scattering Solution Example

First order scattering usually implies $\tau^* \ll 1$, so solution simplifies to

$$I_1^{\uparrow}(\mu, \phi) \; = S_0 \frac{\omega P(\Theta)}{4\pi} \; \frac{\tau^*}{\mu}$$

Molecular Rayleigh scattering at wavelength $\lambda = 0.7\ \mu$m.
Optical depth from molecular scattering formula is $\tau_{mol} = 0.037$.
TOA solar flux at $\lambda = 0.7\ \mu$m is $S_0 = 1400$ W m^{-2} μm^{-1}.
Solar geometry: $\theta_0 = 30°, \phi_0 = 180°$ $(\mu_0 = 0.866)$
Viewing geometry: $\theta = 60°, \phi = 0°$ $(\mu = 0.5)$.
Scattering angle is therefore $\Theta = 90°$. Rayleigh phase function is

$$P(\Theta) = \frac{3}{4}(1 + \cos^2 \Theta) \; = 3/4$$

First order solution is then

$$I_1^{\uparrow}(\mu, \phi) = (1400 \text{ W m}^{-2} \mu\text{m}^{-1}) \frac{0.75}{4\pi} \frac{0.037}{0.5} \quad = 6.2 \text{ W m}^{-2} \text{sr}^{-1} \mu\text{m}^{-1}$$

4. Legendre polynomial expansion of the scattering phase function.

The phase function may be numerically expanded in Legendre polynomials with a finite number of terms N as

$$P(\cos\Theta) = \sum_{l=0}^{N} \varpi_l^* P_l(\cos\Theta) \qquad [17.14]$$

where Θ is the scattering angle

$$cos(\Theta) = cos(\theta')cos(\theta) + sin(\theta')sin(\theta)\ cos(\varphi'-\varphi) = \mu'\mu + (1-\mu'^2)^{1/2}(1-\mu^2)^{1/2}\ cos(\varphi'-\varphi)$$

and ϖ_l^* is the expansion coefficients expressed as

$$\varpi_l^* = \frac{2l+1}{2}\int_{-1}^{1} P(\cos\ \Theta) P_l(\cos\ \Theta) d\ \cos(\ \Theta)\ ,\ l{=}0,\ 1,\ldots,N \qquad [17.15]$$

where: $\mathcal{P}_0 = 1 \quad \mathcal{P}_1 = x \quad \mathcal{P}_2 = (3x^2-1)/2 \qquad \mathcal{P}_n(1) = 1$

NOTE: Orthogonal properties of the Legendre polynomials:

$$\int_{-1}^{1} P_k(\cos\Theta) P_l(\cos\Theta) d\cos(\Theta) = 0\ for\ l \neq k$$

$$\int_{-1}^{1} P_k(\cos\Theta) P_l(\cos\Theta) d\cos(\Theta) = \frac{2}{2l+1}\ for\ l = k$$

Eq.[17.14] can be expressed in the terms of associated Legendre polynomials

$$P(\mu,\varphi,\mu',\varphi') = \sum_{m=0}^{N} \sum_{l=m}^{N} \varpi_l^m P_l^m(\mu) P_l^m(\mu') \cos(m(\varphi'-\varphi)) \qquad [17.16]$$

where

$$\varpi_l^m = (2-\delta_{0,m})\varpi_l^* \frac{(l-m)!}{(l+m)!} \qquad \boldsymbol{l = m,\ldots,N;} \qquad 0 \le m \le N$$

and $\delta_{0,m}$ is the Kronecker delta: $\delta_{0,m} = 1$ for m=0 and otherwise $\delta_{0,m} = 0$.

In similar manner, we may expand the diffuse intensity in the cosine series

$$I(\tau,\mu,\varphi) = \sum_{m=0}^{N} I^m(\tau,\mu) \cos(m(\varphi_0-\varphi)) \qquad [17.17]$$

Using Eqs.[17.16] and [17.17] and the orthogonality of the associated Legendre polynomials, the equation of the radiative transfer for the diffuse intensity (Eq.[17.7]) splits into (N+1) independent equations in the form

$$\mu \frac{dI^m(\tau,\mu)}{d\tau} = I^m(\tau,\mu) - (1+\delta_{0,m})\frac{\omega_0}{4}\sum_{l=m}^{N} \varpi_l^m P_l^m(\mu)\int_{-1}^{1} P_l^m(\mu') I^m(\tau,\mu') d\mu' - $$
$$- \frac{\omega_0}{4\pi}\sum_{l=m}^{N} \varpi_l^m P_l^m(\mu) P_l^m(-\mu_o) F_0 \exp(-\tau/\mu_0) \qquad [17.18]$$

m=0 => azimuthal independent case:

From Eq.[17.16], the azimuth-independent phase function (defined by Eq.[17.6]) can be expressed as

$$P(\mu,\mu') = \sum_{l=0}^{N} \varpi_l P_l(\mu) P_l(\mu') \qquad [17.19]$$

For this case Eq.[17.18] simplifies to (omitting the superscript 0 for m=0)

$$\boxed{\begin{aligned} &\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2} \sum_{l=0}^{N} \varpi_l^* P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau,\mu') d\mu' - \\ &- \frac{\omega_0}{4\pi} \sum_{l=0}^{N} \varpi_l^* P_l(\mu) P_l(-\mu_o) F_0 \exp(-\tau/\mu_0) \end{aligned}} \qquad [17.20]$$

Phase Function Examples

Asymmetry parameter - measures degree of forward scattering

$$g = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\cos\Theta d\cos\Theta \ = \omega_1/3$$

Rayleigh phase function:

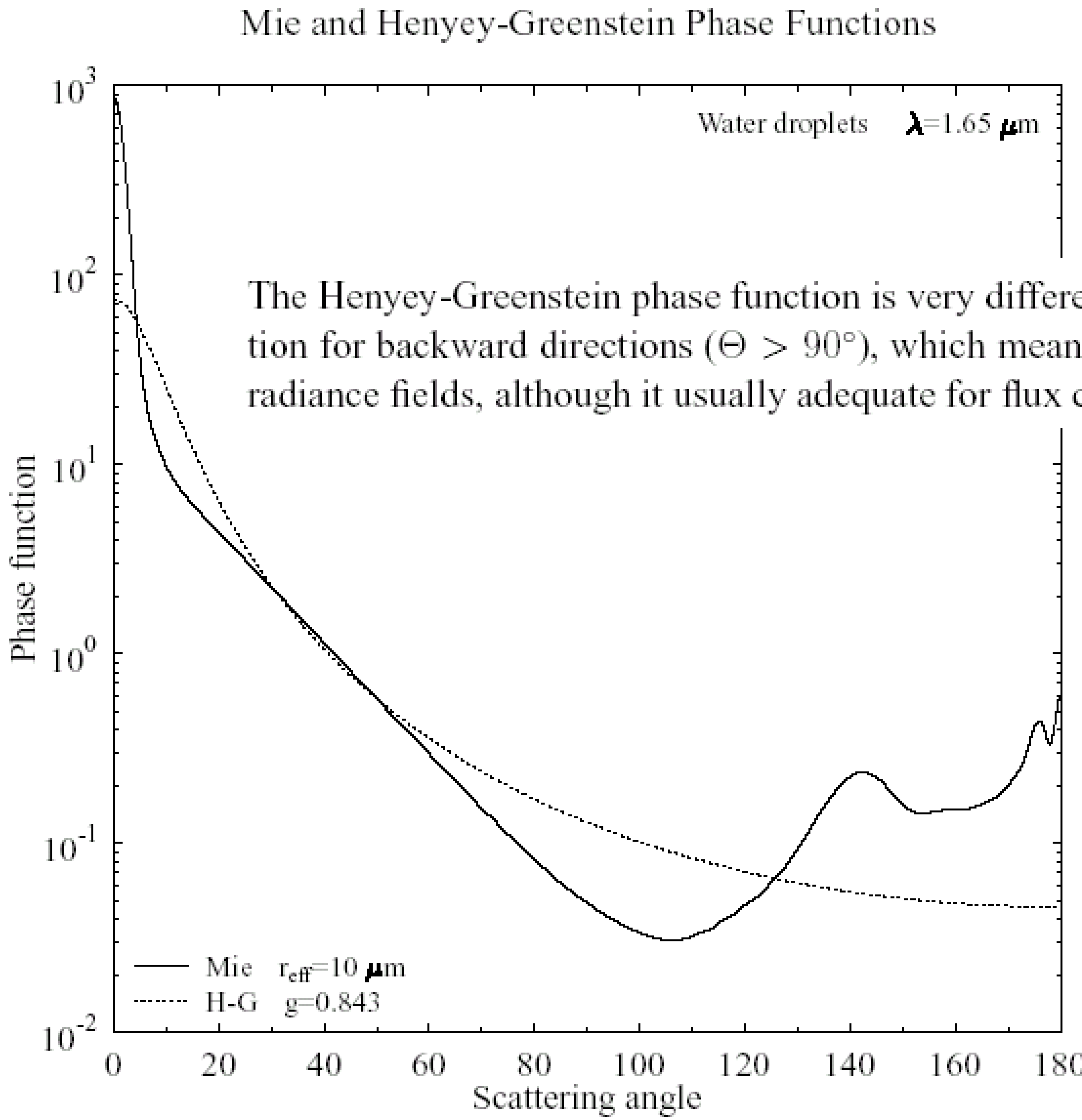
$$\omega_0 = 1 \quad \omega_1 = 0 \quad \omega_2 = 1/2 \quad \omega_l = 0 \ \ l > 2$$

Henyey-Greenstein phase function - often used surrogate for Mie

$$P_{HG}(\Theta) = \frac{1-g^2}{(1+g^2-2g\cos\Theta)^{3/2}}$$

H-G phase function in forward direction: $P_{HG}(0^\circ) = (1+g)/(1-g)^2$.
H-G function in backward direction: $P_{HG}(180^\circ) = (1-g)/(1+g)^2$.
H-G phase function in Legendre polynomials:

$$\omega_l = (2l+1)g^l$$



The Henyey-Greenstein phase function is very different from the Mie phase function for backward directions ($\Theta > 90°$), which means that it is not equivalent for radiance fields, although it usually adequate for flux calculations.

Comparison of Mie and Henyey-Greenstein phase function with same asymmetry parameter g.